The Exact-Numbers Idea: Children's Understanding of Cardinality and Equinumerosity

Barbara W. Sarnecka, Charles E. Wright

University of California-Irvine

The Exact-Numbers Idea: Children's Understanding of Cardinality and Equinumerosity

What does it mean to say that a child understands numbers? There are many early milestones in number learning, and parents sometimes say that a toddler who can count to five or ten 'knows' those numbers. Similarly, young children in literate environments learn to identify the written digits 0-9 along with letters of the alphabet and thus, in a sense, 'know' the numbers. But what does it mean to *understand* numbers, in some important conceptual way? One operational definition comes from Piaget (1952). In that tradition, children understand numbers when they pass the conservation-of-number task, around age five or six. For Piaget, the key number concept is *equinumerosity* (sometimes called *exact equality*)—the idea that two sets have the same number of items, if and only if their members can be placed in perfect one-to-one correspondence (Frege, 1884/1980). The child's understanding of equinumerosity is what the conservation task is supposed to measure.

A different operational definition of number knowledge comes from more recent, 'bootstrapping' theories of number development (e.g., Carey, 2009, Hurford, 1987; Klahr & Wallace, 1976). In this literature, children are said to understand numbers when they apply the cardinality principle of counting (Gelman & Gallistel, 1978) on the Give-N task (Condry & Spelke, 2008; Le Corre, Van de Walle, Brannon & Carey, 2007; Sarnecka & Lee, 2009; Wynn, 1992.) Here, the key concept is *succession* (often called the *successor principle* or *successor function*)—the idea that each number is generated by adding one to the previous number (Dedekind, 1872/1901, 1888/1901). The child's understanding of succession is what the Give-N task is supposed to measure, as it has been argued that only CP-knowers (short for *cardinality-principle knowers,* children who apply the cardinality principle correctly on the Give-N task) understand how counting instantiates the successor principle (Sarnecka & Carey, 2008).

Integrating these literatures, Izard and colleagues identified equinumerosity and succession as “two key concepts on the path towards understanding exact numbers” (Izard, Pica, Spelke & Dehaene, 2008). But how do these concepts interact in development? Our proposal here is that equinumerosity and succession are indeed two aspects of the same, *exact-numbers* concept, and that children either understand both (i.e., they have the exact-numbers concept) or neither (i.e., they don’t yet have the exact-numbers concept).

Note that this conclusion is far from obvious. Children were traditionally credited with the equinumerosity concept when they passed Piaget’s conservation-of-number task, at age 5 or 6. But they are credited with understanding succession much earlier-- when they pass the Give-N task, at age 3 or 4. As Muldoon et al. (2009) noted,

> “The developmental puzzle is that up to the age of six, even some two years after they have mastered procedural counting, many children have yet to grasp that two sets with the same cardinal number must, by virtue of logical necessity, be equivalent, and that sets with different cardinals must by the same logic be numerically different.” (p. 203-204).

Thus, in order to argue that the child’s knowledge of equinumerosity and succession are two faces of the same ‘exact numbers’ idea, we have to argue for at least one of them to be measured differently. What we will argue is that Piaget’s litmus test for equinumerosity (the conservation-of-number task) underestimated children’s knowledge because it asked children about the abstract entity *number*, rather than about particular numbers such as *five* and *six*. (In other words, Piaget asked questions such as, “Are there the same number of flowers and vases?” rather than, “There are five flowers. Are there five vases, or six?”)

Piaget did this deliberately, because he was interested in abstract and explicit knowledge.

But if the bootstrapping accounts are correct, then children develop the exact-numbers concept in the context of particular number words, such as *five* and *six*. The counting list (*one*, *two*, *three*, etc.) is learned as a placeholder structure (something like the chant *eenie*, *meenie*, *minie*, *mo*), with little or no numerical content. The child acquires the deep numerical concepts (e.g., cardinality, equinumerosity, succession) during the process of assigning (or constructing, or discovering, depending on your theoretical bent) meanings for those number words. This is the process known as *conceptual-role bootstrapping* (Carey, 2009; Block, 1987; Quine, 1960). If children first learn about equinumerosity in the context of particular number words such as *five* and *six*, then measuring equinumerosity knowledge outside the context of particular number words (as Piaget did) may lead us to underestimate the knowledge that children have earlier on.

There are hints of this in the findings reported by Sarnecka and Gelman (2004). That study investigated children's understanding of the *specificity* of number words. This is the idea that every number word picks out a specific, unique numerosity (Wynn, 1990, 1992). Some bootstrapping proposals had claimed that children did not understand this property of number words until they mastered the cardinality principle of counting (as measured by the Give-N task). Sarnecka and Gelman developed three new tasks to measure children's knowledge of specificity, two of which children passed before understanding the cardinality principle. Thus, Sarnecka and Gelman concluded that children understood specificity before cardinality.

The present paper revisits the third task—the one children failed until they understood cardinality. This was the 'Compare-Sets' task. In it, children were shown two pictures, representing the snacks given to a pair of animals. The pictures were either identical or differed by one item. The children were told how many items one set had, and then were asked about the other set (e.g., "Frog has five peaches. Does Lion have five, or six?")

The authors intended the Compare-Sets task to measure the child's knowledge that number words are specific. When non-CP-knowers (children who do not yet understand cardinality as measured by the Give-N task) passed two other 'specificity' tasks but failed Compare-Sets, the authors concluded that the task was simply too difficult, and predicted that if the procedural demands could be reduced, the performance gap between CP-knowers and non-CP-knowers would disappear.

The present work tests that prediction, and concludes that it was wrong. A new, simplified version of the Compare-Sets task actually makes the performance gap between CP-knowers and non-CP-knowers even more obvious. In light of this finding, we revisit the question of what the Compare-Sets task actually measures, and argue for an answer that was not considered in the 2004 study: That the task does not primarily measure the child's knowledge of *specificity*, but of *equinumerosity*. So while children may indeed see number words as specific, they do not understand equinumerosity (as the basis for two sets having the same number word) until they become CP-knowers.

Earlier studies have reported findings that are consistent with this possibility, although none have tested it directly. For example, Sophian (1988) presented three- and four-year-olds with two sets of objects (e.g., a group of jars arranged in a circle, with a pile of spoons in the middle). In half the trials, children were told the number of each set, and then asked about their correspondence. For example, "There are *n* jars. There are *m* spoons. Can every jar have its own spoon?" In the other trials, children were told about the correspondence and given the number of one set, and then were asked about the number of the other set. For example, "Every jar has its own spoon. There are *n* jars. Are there *n* spoons?" Sophian reported that about 30-40% of three-year-olds, and 70- 75% of four-year-olds, succeeded on both types of trial. This is approximately the proportion (of Sophian's relatively high-SES sample) that we would expect to be CP-knowers if they were tested on the Give-N task.

Frydman and Bryant (1988) reported something similar. In that study, four-year- olds were asked to divide ('share out') a set fairly, and to count one of the resulting portions. Having done that, many of the four-year-olds were able to infer the number of another, uncounted portion. (See Izard et al, 2008 for a related finding.)

The present study revisits the Compare-Sets task and tests Sarnecka and Gelman's (2004) explanation for the performance gap between CP-knowers and other children (i.e., that the task was too procedurally difficult). A simplified version of the task greatly reduces the burden on attention and memory by leaving the sets visible the whole time.[1] But contrary to Sarnecka and Gelman's prediction, simplifying the task does not eliminate the performance gap between CP-knowers and other children.

In light of these findings, we reconsider how this task should be interpreted. We suggest that the gap in performance between CP-knowers and non-CP-knowers may reflect CP-knowers' understanding of equinumerosity—and that equinumerosity itself may be (along with understanding of the cardinality principle and the successor function) a manifestation of a broad conceptual achievement: The 'exact numbers' idea.

Method

Participants

Participants included 51 children (30 girls, 21 boys). Their ages ranged from 2 years, 7 months to 4 years, 1 month (mean age 3;4). All children were monolingual speakers of English. Children were recruited by mail and phone using public birth records in the greater Boston area, and were tested at a university child development laboratory in Cambridge, Massachusetts.

Parents who brought their children in for testing received reimbursement for their travel expenses and a token gift for their child. No questions were asked about socio-economic status, race, or ethnicity, but participants were presumably representative of the upper-middle SES, predominantly white and Asian communities in which they lived.

Procedure

Give-N task. The purpose of this task was to determine the child's number-knower level (i.e., to determine which exact number-word meanings the child knew) and specifically to determine whether the child understood the cardinality principle of counting. Materials included a stuffed animal (e.g., a bunny, approx. 20 cm high), a plastic plate (approx. 11cm in diameter), and a set of 15 plastic counters (e.g., apples, each approx. 3 cm in diameter). The experimenter began the task by placing the animal on the table and saying, (e.g.) "In this game, we give things to the bunny, like this . . ." (here the experimenter mimed placing something on the plate and sliding the plate across the table to the animal). The experimenter then placed a bowl of 15 apples on the table in front of the child and said, "Can you give the bunny one?" After the child put one or more apples on the plate and slid the plate over to the animal, the experimenter asked a single follow-up question, which repeated the original number word asked for (e.g., "Is that one?") If the child said "yes," then the experimenter said, "Thank you!" and placed the apple(s) back in the bowl. If the child said "no," then the experimenter repeated the original request.

The child was always asked for 1 on the first trial, and for 3 on the second trial. If the child succeeded on both of those trials, the third request was for 5. Otherwise, the third request was for 2. Further requests depended on the child's answers: If a child succeeded at giving some number N, the next request was for N+1; the highest number requested was 6. If the child failed at giving N, the next request was for N-1; the lowest number requested was 1. The task ended when the child had a least 67% successes (with a minimum of two trials) at a given number N, and at least 67% failures (with a minimum of two trials) at N+1. This pattern was the basis for sorting into number-knower levels: Children who succeeded at 1 (but failed at 2 and higher) were called one-knowers; children who succeeded 1 and 2 (but failed at 3 and higher) were called three-knowers, and so forth. Children who were able to generate all set sizes up to and including 6 were called cardinality-principle-knowers. Failures were counted against both numbers involved. For example, if a child gave 4 apples when asked for “two,” that was counted as failure on both “two” and “four.” (For further discussion of number-knower levels and Give-N as a diagnostic task, see Le Corre, et al., 2006; Le Corre & Carey, 2007; Lee & Sarnecka, 2010; Sarnecka & Carey, 2008; Sarnecka & Lee, 2009; Wynn, 1990, 1992.)

Compare-Sets task. The purpose of this task was to test whether children could extend a number word from one set to another on the basis of one-to-one correspondence between the sets. It is based on the task used by Sarnecka & Gelman (2004). Materials for this task included two stuffed animals (a frog and a lion) and 8 pairs of picture cards, depicting the animals’ snacks. Each card showed a homogeneous row of 5 or 6 food items (e.g., peaches). Each pair of cards was either identical (e.g., 5 peaches and 5 peaches) or differed by one item (e.g., 6 cupcakes and 5 cupcakes). When the sets were different, there was an empty circle at one end of the row, highlighting the place where one item was missing.

The experimenter introduced the task in the following way. “This is a story about when Frog and Lion came to my house, and I gave them some snacks. I tried to make their snacks just the same, because they like their snacks to be the same. But sometimes I made a mistake, and their snacks were not the same. The first snack I gave them was peaches . . . ”

Here, the experimenter placed the first pair of cards on the table, one in front of each animal, and asked the first control question, "Are their snacks just the same, or did I make a mistake?" Trials where children answered this question incorrectly were excluded from the analysis. The vast majority of 'incorrect' answers occurred on the first trial where the sets differed, because the child often said that the snacks were 'the same,' meaning that they were the same kind of food. (E.g., the child often said something like "Yes, they both got peaches.") In this case, the experimenter emphasized the discrepancy by saying, e.g., "Well, they both got peaches but . . . oh no! Look! I forgot to put a peach there! Doh! (slapping forehead) That's not right! I made a mistake! I'm so silly!" etc.

After the child's attention had been drawn to the same-ness or difference of the two sets, the experimenter either removed the cards ('hidden' trials) or left them sitting in full view ('visible' trials) and asked the test question, which gave the child the number of one set and asked about the other. In the 'hidden' trials, the question was of the form, "Frog had five cupcakes. Did Lion have five, or six?" In the 'visible' trials, the question was of the form, "This (pointing to Frog's snack) is six peaches. Is this (pointing to Lion's snack) five, or six?"

Children kept their hands in their lap, and did not count the items on the cards. (In the 'hidden' trials, the cards were removed from view before the test question, so there was nothing to count anyway.) On 'visible' trials, if a child made any move to count (e.g., by pointing to an item), the experimenter removed the cards from view and said, "This isn't a counting game. You can just guess" and then waited for the child to return hands to lap before laying the cards back on the table. Such exchanges were rare, because children rarely made any attempt to count the items.

On the 'hidden' trials, the test question was followed by a final control question, "And were their snacks just the same, or did I make a mistake?" Trials where children failed this final control question were also excluded from the analysis.

Each child completed a block of four ‘visible’ trials and a block of four ‘hidden’ trials, for a total of eight trials. Within each block, the set sizes given to Frog/Lion were 5/5, 6/6, 5/6 and 6/5. Order of blocks, and order of trials within each block, were counterbalanced across subjects.

Data Analysis. Responses were binary (correct/incorrect) and each child could contribute up to 8 valid responses, one for each trial. A common way to analyze such data is to collapse across the levels of some factors (e.g., to add up each child’s responses, creating a score of 0-4 for visible trials and a score of 0-4 for hidden trials). However, in this case we chose not to collapse across different trial types because we did not want to assume that any factors were unimportant. Instead, we analyzed these data by fitting generalized, linear, mixed-effects models (McCulloch, 2003) with a logit link function. In the interest of clarity, our presentation of these results will only include the details of the fitting process where those details are critical to understanding or evaluating the results. The actual fits were done using the lme4 (Bates, 2005) package in R (version 2.10.1; R Development Core Team, 2006).

Results

Give-N task. Based on their performance in the Give-N task, children were identified as either cardinality-principle-knowers (CP-knowers, n=22) or non-cardinality-principle-knowers (non-CP-knowers, n=29). Among the non-CP-knowers, there were 5 pre-number-knowers, 5 one-knowers, 9 two-knowers, 7 three-knowers, and 3 four-knowers. There was a strong relationship between age and knower level, $r(51) = .66$, $p = .000$, reflecting the fact that older children knew more than younger children. There was no evidence of differences in the knower levels of boys versus girls, $F(1,49) = 0.713$. Except where noted, the non-CP-knowers were collapsed into a single group in the analyses reported below.

Compare-Sets task. The initial control question, “Are their snacks just the same, or did I make a mistake?” was asked before the test question on all trials. On the ‘hidden’ trials only, the control question was repeated again after the test question, to check that the child still remembered whether the sets had been identical. As described in the method section above, children often misunderstood the control question at first, taking it to mean ‘same type of food’ rather than ‘same amount of food.’ All together, one or both control questions were answered incorrectly on 32% of trials. There was a statistically reliable tendency for the same children to miss control questions in both the hidden and visible conditions, $r(51) = .311$, $p = .027$. There were just three instances in which a child answered the control question correctly and then refused to answer the test question. All subsequent analyses excluded trials on which the child either failed to answer one or both control questions correctly, or did not answer the test question (e.g., trials that were not completed because the child decided to quit playing).

These exclusions eliminated one child from the data set: a female two-knower, age 3;1. We considered also dropping the data from four other children. After the exclusions, these children had no responses on all four trials in either the ‘hidden’ or ‘visible’ block (two children each). However, the data from these four children was retained after we determined that this did not qualitatively alter any of our conclusions. After these exclusions there were 273 responses in the data.

The analysis of the remaining data from the Frog-and-Lion task focused on the effects of five factors. ‘Participants’ was the one random factor. There was one 2-level, between-participant, fixed effect: ‘CP-knower status’ (CP-knower/non-CP-knower). And there were three 2-level, within-participant, fixed effects: ‘visibility’ (hidden/visible), ‘N-first’ (5/6 objects in the first set presented), and ‘same-different' (identical sets/discrepant sets). The model based on these factors that best fit the data was one that included the main effect of CP-knower status (z = 4.033, p = .000), and the main effect of visibility (z = 2.044, p = .041), and that allowed the variance of the random effect of participants to differ across the levels of the visibility factor. The interaction of these two factors was not statistically reliable (z = 1.587, p = 0.113). Collapsing over the visibility factor, children in the non-CP-knower group performed better than chance (z = 6.149, p = .000).

However, being a CP-knower increased the average probability of responding of responding correctly from .59 to .85. Collapsing over CP-knower status, having the sets visible rather than hidden increased probability of responding correctly from .67 to .80. This final model was simpler because none of the potential terms involving either of the presentation variables (N-first or same-different) did much to improve the fit of the model. In other words, it made little difference whether the first set size presented was 5 or 6; nor did it matter much whether the sets were identical or different. For these terms the most significant had z = -1.367, p = 0.172 and most were substantially less important.

We also re-ran the analysis to include the 70 trials on which the child missed the first control question, because Sarnecka and Gelman (2004) did include such trials in their analysis. (‘Hidden’ trials on which the child missed the second control question were still excluded.) Results were similar to the first analysis, showing main effects of CP-knower status (z = 4.962, p = .000) and visibility (z = 2.350, p = .019). However, in this analysis the interaction of these two factors was also statistically significant (z = 2.046, p = .041), meaning that having the sets remain visible was more helpful to CP-knowers than to non-CP-knowers.

Analysis of age effects. Before accepting that this model provided an appropriate summary of these data, we felt it was important to explore two plausible alternatives. The first is that CP-knower status and success in this task both may reflect a developing maturity that can be indexed by age. Certainly, as reported in the previous section, number-knower level in these data was strongly correlated with age (see Fig. 1a). However, when the model was expanded to include linear, quadratic, and cubic age terms, the fit was improved no more than if these had been random predictors (□2(3) = 2.097, p = .553), and the size of the coefficient in the model associated with CP-knower status was attenuated by less than 1% and remained statistically significant (z = 3.770, p = .000). In other words, despite the correlation between knower level and age, it was CP-knower-status (and not age) that predicted success on the task (see Fig. 1b.)

Analysis of differences among the non-CP-knower levels. The second alternative explored whether differences in number-knower level, other than the distinction of CP- knower/non-CP-knower, explained any variation in performance. Including number-knower level (i.e., pre-knower, one-knower, two-knower, three-knower, or four-knower) instead of CP- knower-status (i.e., CP-knower or non-CP-knower) as a factor in the model did not substantially improve the model fit (□2(4) = 6.765, p = .149). As it happened, the only non-CP-knower level with performance that differed reliably from the overall average was three-knowers, who did significantly worse than the average (z = -2.827, p = .005; lacking any principled explanation for this anomaly, we assume that it was a fluke.) In the model that included number- knower level as a factor, there was still a reliable difference between CP-knowers and the overall average, z = 3.866, p = .000 (see Fig. 2).

Discussion

These results suggest a number of things. First, Sarnecka and Gelman’s (2004) interpretation of the Compare-Sets task was at least partially flawed. In that paper, the task was seen as a way of measuring the child's knowledge of *specificity* (i.e., the idea that each number word picks out a specific, unique numerosity). But if one accepts that children see number words as specific before they figure out the cardinality principle of counting, then there is a puzzle: Why do CP-knowers (i.e., children who have already figured out the cardinality principle) perform so much better than non-CP-knowers on the Compare-Sets task?

Sarnecka and Gelman's (2004) answer was that the task was too procedurally difficult, too taxing on attention and memory. According to that explanation, if the task could be made simpler, then non-CP-knowers should do fine. But the present data do not bear out this prediction. We gave children the original version of the task (where the sets were hidden) as well as a new, simpler version of the task where the sets remained visible the whole time. If Sarnecka and Gelman's prediction had been correct, then non-CP-knowers should have performed substantially better on the simplified, "visible" version of the task than on the older "hidden" version—but that was not the case. In fact, the one analysis that showed an interaction found that keeping the sets visible was actually more helpful to the CP-knowers than to the non-CP-knowers.

The data present two patterns that seem to require explanation: First, the non-CP-knowers (as a group) performed slightly better than chance. Second and more striking, the CP-knowers performed much better than the non-CP-knowers, not only at the group level but at the individual level. In fact, every single CP-knower outperformed every single non-CP-knower (see Figs. 1b and 2).

Any explanation for these patterns is necessarily speculative, but the lack of overlap in performance by CP-knowers and non-CP-knowers seems consistent with the possibility that the two groups are using different strategies on the task. (Or that the CP-knowers have a strategy, and the non-CP-knowers don't have much of one.)

For example, the non-CP-knowers could perform slightly better than chance because some implicit pragmatic bias led them to repeat the same word when the sets were termed "the same," and to choose the other word when the sets were termed "not the same." Such a rule need not be specific to number words, and could apply in the absence of any conceptual understanding of what makes two sets numerically "the same" – that is, without the child having any understanding of equinumerosity. (Of course a child who represented this rule explicitly, and applied it consistently, would get every trial correct. Because no non-CP-knower approached perfect performance, it might make more sense to think of such a pragmatic constraint as operating subtly and implicitly on children's behavior.

The more interesting question is, why do CP-knowers (and only CP-knowers) perform so well on this task? What number knowledge do they have, that non-CP-knowers lack? Obviously, CP-knowers (by definition) understand how counting works. But children were not allowed to count the items in the Compare-Sets task, so counting skill alone cannot directly explain the CP-knowers' success.

It has been argued that only CP-knowers understand succession—that is, they understand how the successor function generates a set size to go with each number word (Sarnecka & Carey, 2008). But this is still an unsatisfying explanation for the present results, because the Compare-Sets task doesn't directly measure understanding of succession. (E.g., it doesn't measure the child's knowledge that the set is increased by exactly one item with each word in the list).

What the task does directly measure is knowledge of equinumerosity—the idea that two sets whose members form a one-to-one correspondence have the same number. Equinumerosity may be (like cardinality and succession) a concept that CP-knowers have and non-CP-knowers lack. If so, then the conceptual achievement that has long been called the *cardinality-principle induction* might be better termed the *cardinality-principle-successor-function-equinumerosity* induction, an unwieldy term indeed. Following Izard et al. (2008), we prefer to use a simpler term: The ‘exact numbers’ idea.

What does it mean, after all, to understand cardinality and succession and equinumerosity? It means understanding what counting is, how counting and number words relate to set sizes, what it means for two sets to have the same number . . . all of these are inferences licensed by a particular notion of exact numbers. We believe that it is this notion, this ‘exact numbers’ idea, that separates CP-knowers from non-CP-knowers and explains their differing performance on Compare-Sets, and many other number tasks.

There is another possibility to consider. CP-knowers might succeed on the Compare-Sets task, not because they understand equinumerosity, but because they have mapped the number words “five” and “six” to quantity representations in the innate approximate number system. Le Corre and Carey (2007) showed that non-CP-knowers do not have such mappings, and that children construct them several months after making the CP-induction. If the CP-knowers (and only the CP-knowers) were able to estimate five and six items without counting, then they might be able to answer the question “Does Lion have five, or six?” simply by looking at (or remembering) Lion’s snack and estimating how many items were in it. This might also explain why keeping the sets visible was more helpful to CP-knowers than to non-CP-knowers.

However, there are two problems with this explanation. First, it would require that *all* the CP-knowers in our study had mapped “five” and “six” to the ANS, because all the CP-knowers performed quite well on the task. But Le Corre and Carey (2007) found that children do not construct such mappings until some six months after making the CP-induction. This would lead us to expect that at least some of the CP-knowers (i.e., the ones who had not been CP-knowers for very long) would still be what Le Corre and Carey called “non-mappers.” The fact that in our study, *every* CP-knower performed well suggests that their performance was tied more directly to the CP-induction itself. The other problem with the ANS-mapping explanation is that the CP-knowers performance just seems more accurate than would be expected under an estimation account. Even if all the CP-knowers in our sample had constructed ANS mappings for the number words, it seems unlikely that these mappings would be precise enough to allow the children to correctly discriminate 5 from 6 some 85% of the time. For these reasons, we tend to favor the equinumerosity explanation over the ANS-mapping explanation for now.

If we accept that children’s understanding of equinumerosity, succession and cardinality are all bound together in one ‘exact numbers’ idea, and that this idea is acquired after lengthy period of partial but incomplete knowledge of counting and number, what are the implications of that? For theoretical debates, the implication would seem to be strong support for the ‘bootstrapping’ approaches (e.g., Carey, 2009). These approaches have long argued that when children figure out the cardinal principle of counting, they are really acquiring a deep conceptual understanding of what numbers are. The present findings provide support for that view, by showing yet another number task on which children’s performance is strongly correlated with their knowledge of cardinality.

But to us, the most important implication of the present work is educational. One of the main goals (perhaps *the* main goal) of preschool math education should be to make sure that all children have this ‘exact numbers’ idea (i.e., make sure that all children are CP-knowers) before they start kindergarten. Children who lack concepts of cardinality, succession and equinumerosity really do not know what numbers are. Without that understanding, they cannot make sense of number operations, greater-than/less-than comparisons, or other content in the school math curriculum. We hope that the present work will help to clarify the importance of the ‘exact numbers’ idea as a conceptual achievement in early childhood, so that all children can start school with the knowledge they need to succeed.

References

Bates, D. M. (2005). Fitting Linear Mixed Models in R. *R News, 5*, 27-30.

Block, N.J. (1986). Advertisement for a semantics for psychology. In P.A. French (Ed.), *Midwest studies in philosophy* (pp. 615-678). Minneapolis: University of Minnesota.

Carey, S. (2009). *The origin of concepts*. New York: Oxford University Press.

Condry, K. F., & Spelke, E. S. (2008). The development of language and abstract concepts: The case of natural number. *Journal of Experimental Psychology: General*, *137,* 22-38.

Dedekind, R. (1901). *Essays in the theory of numbers, 1. Continuity of irrational numbers, 2. The nature and meaning of numbers* (W. W. Beman, Trans.). Retrieved from Canadian Libraries Ebook and Texts Archive: http://www.archive.org/details/essaysintheoryof00dedeuoft (Originals published in 1872 and 1888, respectively).

Frege, G. (1980). *The Foundations of Arithmetic: A Logico-Mathematical Enquiry into the Concept of Number* (J. L. Austin, Trans.). Evanston, IL: Northwestern University Press. (Original published 1884.)

Frydman, O. & Bryant, P. (1988). Sharing and understanding of numerical equivalence by young children. *Cognitive Development, 3*, 323-339.

Gelman, R. & Gallistel, C.R. (1978). *The Child's Understanding of Number*. Cambridge, MA: Harvard University Press.

Hurford, J. R. (1987). *Language and number: The emergence of a cognitive system*. Cambridge, MA: Blackwell.

Izard, V., Pica, P., Spelke, E.S., & Dehaene, S. (2008). Exact equality and successor function: Two key concepts on the path towards understanding exact numbers. *Philosophical*

Psychology, 21, 491-505.

Klahr, D., & Wallace, J. G. (1976). *Cognitive development: An information processing view.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Le Corre, M. & Carey, S. (2007). One, two, three, four, nothing more: An investigation of the conceptual sources of the verbal counting principles. *Cognition, 105,* 395-438.

Le Corre, M., Van de Walle, G., Brannon, E.M., and Carey, S. (2006). Re-visiting the competence/performance debate in the acquisition of counting as a representation of the positive integers. *Cognitive Psychology, 52,* 130-169.

Lee, M.D. & Sarnecka, B.W. (2010). A model of knower-level behavior in number-concept development. *Cognitive Science, 34,* 51-67. PMC2836733. doi: 10.1111/j.1551-6709.2009.01063.x

McCulloch, C. E. (2003). Generalized linear mixed models. *NSF-CBMS regional conference series in probability and statistics, v. 7*. Beachwood, Ohio: Institute of Mathematical Statistics.

Muldoon, K., Lewis, C. & Freeman, N. (2009). Why set-comparison is vital in early number learning. *Trends in Cognitive Sciences, 13,* 203-208.

Piaget, J. (1952). *The Child's Conception of Number*. London: Routledge & Kegan Paul Ltd.

Quine, W.V.O. (1960). *Word and object*. Cambridge, MA: MIT Press.

R Development Core Team (2006). *R: A language and environment for statistical computing.* R Foundation for Statistical Computing, Vienna, Austria. Available from http://www.R-project.org.

Sarnecka, B.W. & Carey, S. (2008). How counting represents number: What children must learn and when they learn it. *Cognition, 108,* 662-674.

Sarnecka, B.W. & Gelman, S. (2004). Six does not just mean a lot: Preschoolers see number words as specific. *Cognition*, *92*, 329-352.

Sarnecka, B.W. & Lee, M. D. (2009) Levels of number knowledge during early childhood. *Journal of Experimental Child Psychology, 103,* 325-337. PMC3127737**.** doi:10.1016/j.jecp.2009.02.007

Sophian, C. (1988). Early developments in children's understanding of number: Inferences about numerosity and one-to-one correspondence . *Child Development*, *59*, 1397-1414.

Wynn, K. (1990) Children's understanding of counting. *Cognition*, *36*, 155-193.

Wynn, K. (1992). Children's acquisition of number words and the counting system. *Cognitive Psychology*, *24*, 220-251.

Author Note

Barbara W. Sarnecka and Charles E. Wright, Department of Cognitive Sciences, University of California-Irvine.

This material is based upon work supported by the National Institutes of Health under NICHD R03HD054654 to the first author, and by the National Science Foundation under DRL 0953521 to the first author. Data collection was supported by the National Science Foundation under REC 0337055 to Elizabeth Spelke and Susan Carey. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors, and do not necessarily reflect the views of the National Institutes of Health or the National Science Foundation. We thank the children who participated in this study and their families; thanks also to research assistants Alexandra Cerutti and Jyothi Ramakrishnan for their help with data collection, and to Prof. Jeremy Heis for his insights into the history and philosophy of numbers.

Correspondence concerning this article should be addressed to Barbara Sarnecka, UCI Department of Cognitive Sciences, Irvine, CA 92697-5100. Email: sarnecka@uci.edu

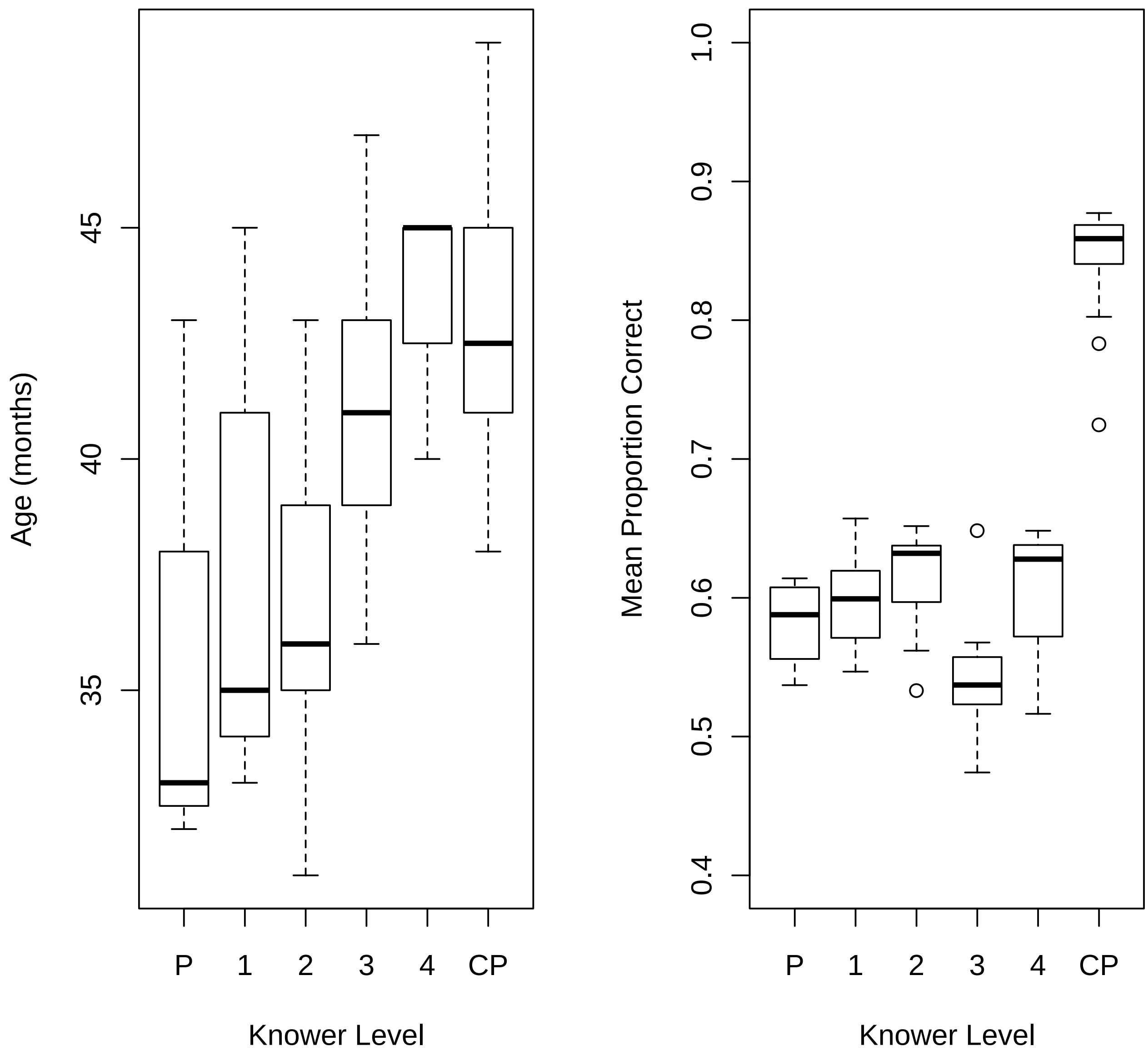
Footnotes

[1] Thanks to Prof. Kirsten Condry for suggesting this way of simplifying the task.

Fig. 1. Box-and-whiskers plots showing the relation between (a) age and knower level, and (b) performance and knower level. Boxes enclose the middle 50% of values; whiskers show the entire range of values; bold line in each box shows the mean value; open circles are outliers.

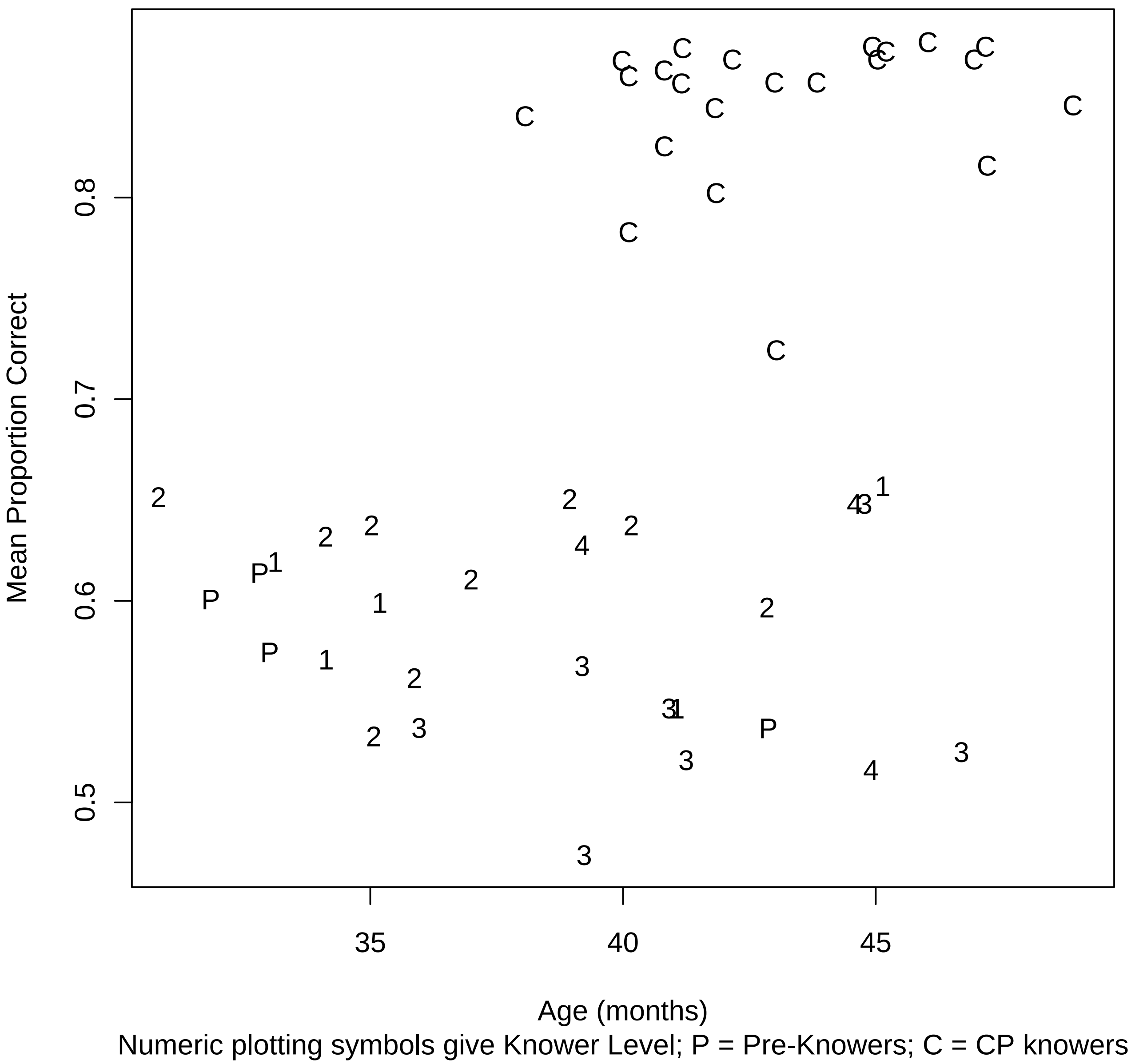
Fig. 2. Scatterplot showing performance of individual participants by age and knower level.

Figure 1



Age (months)
35
40
45
P
1
2
3
4
CP
Knower Level
Mean Proportion Correct
0.4
0.5
0.6
0.7
0.8
0.9
1.0
P
1
2
3
4
CP
Knower Level

Figure 2


Mean Proportion Correct
0.5
0.6
0.7
0.8
35
40
45
Age (months)
Numeric plotting symbols give Knower Level; P = Pre-Knowers; C = CP knowers